**University of Jeddah**
**Faculty of Engineering**
**Department of Electrical & Computer Engineering**

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**Electromagnetic  Fields (ECE 308)**

**Lecture 2 – Vector Analysis - II**

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# Introduction:

When dealing with a scalar physical quantity whose magnitude depends on a single variable, such as the temperature T as a function of height z, the rate of change of T with height can be described by the derivative dT/dz. However, if T is also a function of x and y, its spatial rate of change becomes more difficult to describe because we now have to deal with three separate variables. The differential change in T along x, y and z can be described in terms of the partial derivatives of T with respect to the three coordinate variables. Furthermore, many of the quantities we deal with in electromagnetics are vectors, and therefore both their magnitudes and directions may vary with spatial position. In this respect, we introduce three fundamental operators to describe the differential spatial variations of scalars and vectors, namely: **gradient, divergence and curl**. The gradient operator applies to scalar fields and other two operator applies to vector fields.
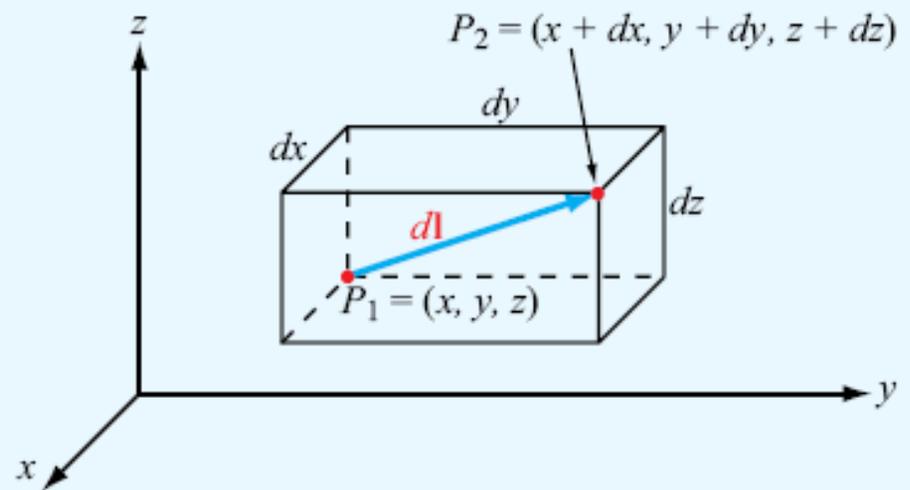
# Gradient of a Scalar Field:

Suppose that $T_1$ = T(x, y, z) is the temperature at point $P_1$=(x, y, z) in some region of space, and $T_2$ = T (x+dx, y+dy, z+dz) is the temperature at point $P_2$ = (x+dx, y+dy, z+dz) as shown in Figure.

-The differential distances dx, dy and dz are the components of the differential distance vector dl is:



Figure 3-19 Differential distance vector dl between points $P_1$ and $P_2$.

$$dl = \hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz.$$

- From differential calculus, temperature difference :

dT = $T_2 - T_1$ is given by $$dT = \frac{\partial T}{\partial x}\, dx + \frac{\partial T}{\partial y}\, dy + \frac{\partial T}{\partial z}\, dz.$$

# Gradient of a Scalar Field:

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, can be rewritten as

$$dT = \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l}$$

$$= \left[ \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \right] \cdot d\mathbf{l}.$$



**Figure 3-19** Differential distance vector $d\mathbf{l}$ between points $P_1$ and $P_2$.

The vector inside the square brackets relates the change in temperature dT to a vector change in direction dl. This vector is called the **gradient of T**, or **grad T**, and denoted by

$$\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$
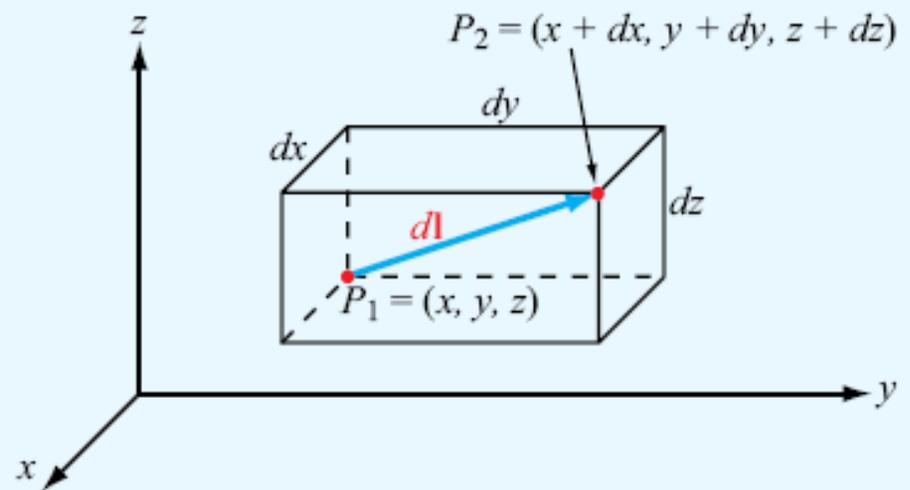
Properties of the Gradient Operator

(1) $\quad \nabla(U + V) = \nabla U + \nabla V,$

(2) $\quad \nabla(UV) = U\,\nabla V + V\,\nabla U,$

(3) $\quad \nabla V^n = n V^{n-1} \nabla V, \qquad$ for any $n$.

# Divergence of a Vector Field:

From the coulomb's law, it is clear that an isolated positive point charge **q** induces an electric field **E** in the space around it, with the direction of **E** being outward away from the charge. Also, the magnitude of **E** is proportional to **q** and decreases with distance **R** from the charges as **$1/R^2$**. In a graphical representation, a vector field is usually represented by a **field lines**, as in fig. The arrowhead denotes the direction of the field at the point where the field line is drawn, and the length of the line provides a qualitative depiction of the field's magnitude.
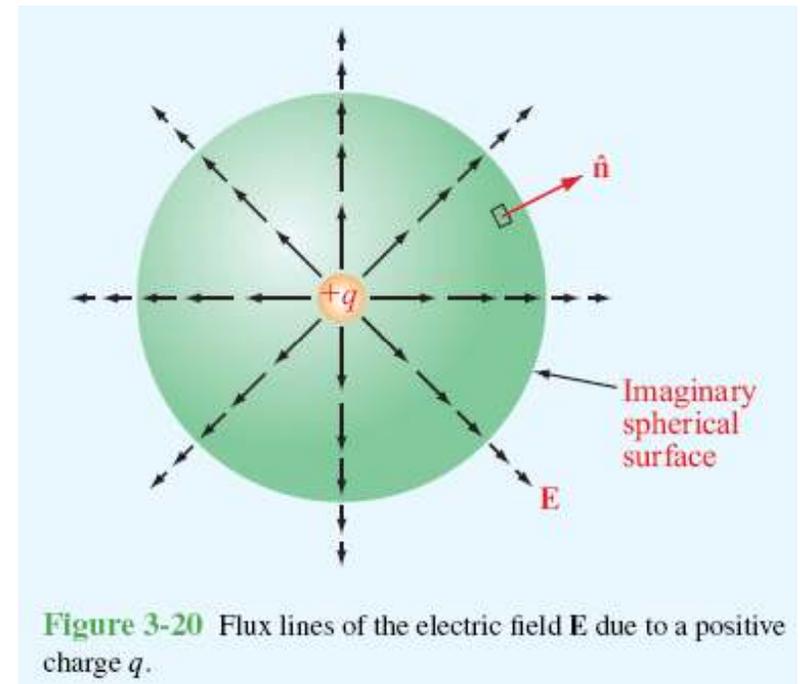


Figure 3-20 Flux lines of the electric field E due to a positive charge $q$.
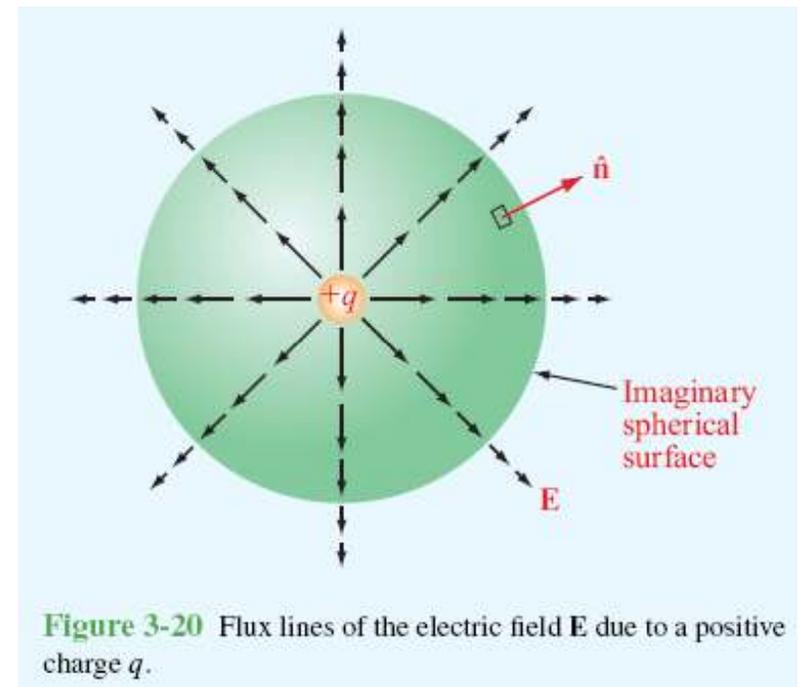
# Divergence of a Vector Field:

At a surface boundary, flux density is defined as the amount of outward flux crossing a **unit surface ds**:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot d\mathbf{s}}{|d\mathbf{s}|} = \frac{\mathbf{E} \cdot \hat{\mathbf{n}} \, ds}{ds} = \mathbf{E} \cdot \hat{\mathbf{n}},$$

The total flux outwardly crossing a closed surface S, such as the enclosed surface of the imaginary sphere is



Figure 3-20 Flux lines of the electric field **E** due to a positive charge $q$.

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}.$$

# Divergence of a Vector Field:

A vector field E (x, y, z) exists in the region of space containing the parallelepiped, and we want to determine the flux of E through its total surface S. So, E is to be defined as:

$$\mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z.$$

The area of the face marked 1 in fig. is $\Delta x$, $\Delta y$, and $\Delta z$ and its unit vector is $\hat{\mathbf{n}}_1 = -\hat{\mathbf{x}}$. Hence, the outward flux $F_1$ through face 1 is:

$$F_1 = \int_{\text{Face 1}} \mathbf{E} \cdot \hat{\mathbf{n}}_1 \, ds$$

$$= \int_{\text{Face 1}} (\hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z) \cdot (-\hat{\mathbf{x}}) \, dy \, dz$$

$$\approx -E_x(1) \, \Delta y \, \Delta z,$$
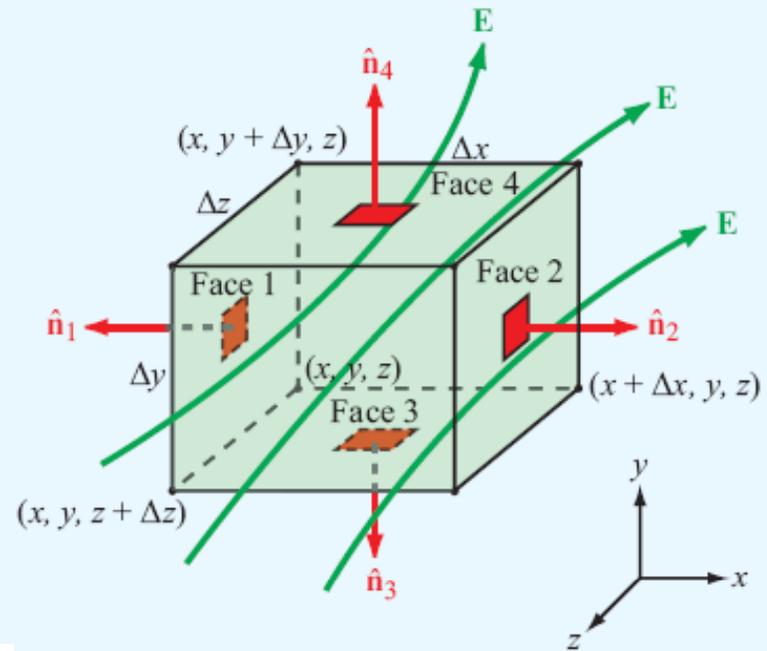


Figure 3-21    Flux lines of a vector field **E** passing through a differential rectangular parallelepiped of volume $\Delta v = \Delta x \, \Delta y \, \Delta z$.

# Divergence of a Vector Field:

Similarly, the flux out of face 2 (with $\hat{n}_2 = \hat{x}$) is: $F_2 = E_x(2)\ \Delta y\ \Delta z,$

Where Ex(2) is the value of Ex the center of face 2. Over a differential separation Δx between the centers of faces 1 and 2, Ex(2) is related to Ex(1) by:

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x}\ \Delta x,$$

Put the value of Ex(2) in F2, getting: $F_2 = \left[ E_x(1) + \frac{\partial E_x}{\partial x}\ \Delta x \right] \Delta y\ \Delta z.$

The sum of the fluxes out of faces 1 and 2 is : $F_1 + F_2 = \frac{\partial E_x}{\partial x}\ \Delta x\ \Delta y\ \Delta z.$

Similarly for 3, 4, 5 and 6, we get:

$$F_3 + F_4 = \frac{\partial E_y}{\partial y}\ \Delta x\ \Delta y\ \Delta z,$$

$$F_5 + F_6 = \frac{\partial E_z}{\partial z}\ \Delta x\ \Delta y\ \Delta z.$$

# Divergence of a Vector Field:

The sum of fluxes F1 through F6 gives the total flux through surface S,

$$\oint_S \mathbf{E} \cdot ds = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

$$= (\text{div } \mathbf{E}) \, \Delta \mathcal{V},$$

where $\Delta \mathcal{V} = \Delta x \, \Delta y \, \Delta z$ and div $\mathbf{E}$ is a scalar function called the *divergence* of $\mathbf{E}$, specified in Cartesian coordinates as

$$\text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} . \qquad (3.94)$$

$$\int_\mathcal{V} \nabla \cdot \mathbf{E} \, d\mathcal{V} = \oint_S \mathbf{E} \cdot ds.$$

**(divergence theorem)**

# Thank you !