Variational Analysis Approach and its Applications to Robotics

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***Abstract*—This paper presents a methodology for variational analysis for the control systems emphasize on Robotics. This is achieved by Lagrangian Mechanics and Variational Collision Integrators. Variational techniques are used to analyze the problem of rigid-body dynamics with impacts. Principles of nonsmooth mechanics for rigid bodies are used in both continuous and discrete time, and provide impact models for a variety of collision behaviors. The discrete Euler-Lagrange (E-L) equations that follow from the discrete time analyses yield variational integration schemes for the different impact models. These discrete E-L equations play a vital role in the method of discrete mechanics and optimal control, which generates locally optimal control policies as the solution to equality constrained nonlinear optimization problems.**

***Index Terms*—Variational Analysis, Discrete Mechanics, Robotics**

I. Introduction

Variational principles play a fundamental role in physics and mechanics. They state that a system adjusts its state always in a manner such that the associated functional is extremal: A soap bubble minimizes the surface area subject to a given volume, an elastic body minimizes the stored elastic energy subject to given boundary conditions, and a temperature-dependent system maximizes entropy subject to a given energy. Thus, one can determine equilibrium states in a canonical way by studying local or global minimizers or maximizers of a functional. Applications are found in many areas include:

- linear and nonlinear elastostatics and contact problems
- microstructures in plasticity and shape memory alloys
- free boundary problems and multiphase problems
- equilibria in reaction-diffusion systems
- ground states in quantum mechanics and density functional theory
- optimization problems
- risk minimization in finance and economy

The calculus of variations provides various strong tools for the solution of partial differential equations and is especially useful in the field of multiscale modeling, e.g. for dimension reduction or for the characterization of microstructures in solids.

It is an exciting time to study robotics since last couple of decades. Driven by nature's examples in animals and humans alike, engineers and roboticists have persistently worked at synthetically capturing the abilities and efficiencies associated with locomotion. Particularly, the last decade has seen a number high profile industry success, notably Sony's Qrio [1] and Honda's Asimo [2], pushing the boundaries of robotic capabilities in walking. At first glance, the performance of these robots can make it seem that the major challenges in the field are foregone. In fact, these robots have given rise to new problems and have motivated new research directions.

The major problem in robotics is contact dynamics, and contact dynamics is governed by *measure differential inclusion*, a general formulation that can directly incorporate impulsive forces and nonsmooth solutions. A measure differential inclusion has the form

$$\frac{dv}{dt} \in F(t,x),\ \frac{dx}{dt}{=}g(t,x,v)$$

where *v(t)* and *x(t)* denote the velocity and the position, F is a set-valued function, and $v(\cdot)$ is required only to have bounded variation. The measure differential inclusion has also been proved to be an excellent mathematical foundation for the study of numerical methods for discontinuous ODEs.

Various other numerical methods for rigid-body systems have been studied extensively in the engineering and mathematics literature. We particularly note the approach that reduces the contact to a *complementarity problem*, a concept frequently used in constrained optimization, to decide at each step which constraints are active. However, most existing practical codes are based on smoothing techniques, a class of methods which use a penalty formulation to regularize the problem. This approach relies on the definition of a proper gap function as a means to detect and penalize the interpenetration; see, for example, [3, 4, 5, 6]. An obvious weakness of the penalty methods is that they cannot handle collision of irregularity shaped bodies (bodies with corners), where neither normals nor gap functions can be defined. An elegant solution to this problem is offered by the *nonsmooth analysis* approach from [7], where new robust contact algorithms are derived using the powerful tools of nonsmooth calculus [8].

Our approach is based on a variational methodology that goes back to [9] which allows the direct handling of the nonsmooth nature of contact problems. We also use a variational approach to develop numerical integrators for nonsmooth rigid-body dynamics. The procedure is based on a discrete E-L principle and automatically generates a symplectic-momentum preserving integrator. To date there have been no symplectic methods for collisions presented, in part due to difficulties with understanding symplecticity in a nonsmooth setting. However, the variational formulation of continuous time nonsmooth systems that we develop here is a key which allows us to understand the geometric structure of the problem, both before and after discretization.

In section II, we first understood continuous model and in section III, discrete model from continuous mode for robot.

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II. Continuous Model

First we consider the system defined by the Lagrangian $L(q, \dot{q}) = \frac{1}{2}\ \dot{q}^T$ M $\dot{q}$- $V(q)$, where M is a mass matrix and V is a potential function. Here $q = (q^1, ...q^n)$ is a vector of configuration variable which lives in the configuration manifold $q \in Q$ and subset $C \subset Q$. We now consider a trajectory $q(t)$ which maps $q : [0, T] \rightarrow Q$ such that $q(t) \in C$, except at a particular time t_i for which $q(t_i) \in \partial C$, where ∂C is a admissible set and we allow the trajectory $q(t)$ to be nonsmooth but still continuous at this time.

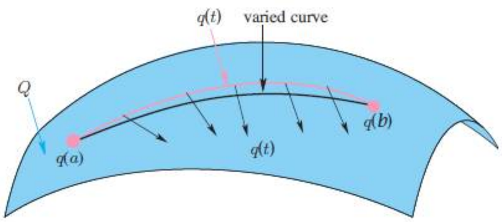

Fig. 1. Continuous Lagrangian w.r.t. E-L Equations

By integrating the Lagrangian along $q(t)$, we get

$$\delta \int_0^T L(q(t), \dot{q}(t))dt$$
$$= \int_0^{t_i} [\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q}]dt + \int_{t_i}^T [\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q}]dt - [L(q, \dot{q}) \cdot \delta t_i]_{t_i^-}^{t_i^+}$$
$$= \int_0^{t_i} [\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})] \cdot \delta q dt + \int_{t_i}^T [\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})] \cdot \delta q dt - [\frac{\partial L}{\partial \dot{q}} \cdot \delta q + L]_{t_i^-}^{t_i^+}$$

where we have used integration by parts and the condition $\delta q(T) = \delta q(0) = 0$. The variations of the action be zero for all δq implies that on the intervals away from t_i the integrand must be zero, giving the well-known Euler-Lagrange equations,

$$\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})) = 0$$

For the particular form of the Lagrangian chosen above, this is simply

$$M\ddot{q} = -\nabla V(q)$$

which is Newton's equation of mass times acceleration equals force, and this equation describes the motion of the system away from impact.

Not only the two integrals in the variational equation be zero, but the jump term at t_i must also be zero, and hence

$$\delta q(t_i) + \dot{q}(t_i) \cdot \delta t_i \in T\partial C$$

which states that the combine variation on the left-hand side must be in the tangent plane to ∂C at the impact point. The space of allowable $\delta q(t_i)$ and δt_i is spanned by the set of $\delta q(t_i) \in T\partial C$ with $\delta t_i = 0$, together with the additional variation $\delta q(t_i) = -\dot{q}(t_i)$ with $\delta t_i = 1$. This gives two relations from variational equations,

$$[\frac{\partial L}{\partial \dot{q}}|_{t_i^+} - \frac{\partial L}{\partial \dot{q}}|_{t_i^-}] \cdot \delta q(t_i) = 0 \text{ for all } \delta q(t_i) \in T\partial C,$$
$$[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L]_{t_i^+} - [\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L]_{t_i^-} = 0$$

when the Lagrangian is of the form kinetic minus potential, as above, these can be written as

$$\dot{q}(t_i^+) - \dot{q}(t_i^-) \in N_C(q_i(t)) \text{ —— (1)}$$
$$E_L t_i^+ - E_L t_i^- = 0 \text{ —— (2)}$$

where the energy is $E_L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} + V(q)$ and $N_C(q)$ is the normal cone to ∂C at q. Equation (1) tell that jump in velocity at the impact point must be orthogonal to the boundary ∂C, while eq. (2) states that energy must be conserved during the impact. Together these two equations constitute a system of n equations which describe the evolution of the system during the collision.

It is well known that the system described by the Euler-Lagrange equations has many special properties. In particular, the flow on state space is symplectic, meaning that it conserves a particular two-form, and if there are symmetry actions on phase space, then there are corresponding conserved quantities of the flow, known as *momentum maps*. All of these geometric properties can be proven directly from the variational principle used above, and so they also hold for nonsmooth systems.

III. Discrete Model

Discrete variational mechanics is based on replacing the position q and velocity $\dot{q}$ with two nearby positions q_0 and q_1 and a timestep h. These positions should be thought of as being two points on a curve at time h apart so that $q_0 \approx q(0)$ and $q_1 \approx q(h)$ for some short curve segment $q(t)$.

Now consider discrete Lagrangian $L_d(q_0, q_1, h)$, which we think of as approximating the action integral along the curve segment between q_0 and q_1. Consider the very simple approximation given by

$$L_d(q_0, q_1, h) = h[(\frac{q_1 - q_0}{h})^T M(\frac{q_1 - q_0}{h}) - V(q_0)]$$

This is simply the rectangle rule applied to approximation the action integral, with the velocity being approximated by the difference operator. Now consider a discrete curve of points $\{q_k\}_{k=0}^N$ in C and corresponding times $t_k = kh$, together with a special impact point $\tilde{q} \in \partial C$ and an impact time $\tilde{t} = \alpha t_{i-1} + (1 - \alpha)t_i$. Here $\alpha \in [0, 1]$ is a parameter which interpolates $\tilde{t}$ with the interval $[t_{i-1}, t_i]$. Given such a discrete trajectory

$$(q_0, t_0), ...(q_{i-1}, t_{i-1}), (\tilde{q}, \tilde{t}), (q_i, t_i), ...(q_N, t_N),$$

We calculate the discrete action along this sequence by summing the discrete Lagrangian on each adjacent pair, with the timestep being the difference between the pair of times. We compute variations of this action sum with respect to variations in the q_k as well as $\tilde{q}$ and α (and hence $\tilde{t}$), with the boundary points q_0 and q_N held fixed. This gives,

$$\delta[\Sigma_{k=0}^{i-2} L_d(q_k, q_{k+1}, h) + L_d(q_{i-1}, \tilde{q}, \alpha h) + L_d(\tilde{q}, q_i, (1 - \alpha)h) + \Sigma_{k=i}^{N-1} L_d(q_k, q_{k+1}, h)]$$
$$= \Sigma_{k=0}^{N-1}[D_1 L_d(q_k, q_{k+1}, h) \cdot \delta q_k + D_2 L_d(q_k, q_{k+1}, h \cdot \delta q_{k+1}]$$
$$= \Sigma_{k=1}^{i-2}[D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) \cdot \delta q_k]$$
$$+ \Sigma_{k=i+1}^{N-1}[D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h)] \cdot \delta q_k$$

$+ [D_2L_d(q_{i-2}, q_{i-1}, h) + D_1L_d(q_{i-1}, \tilde{q}, \alpha h)] \cdot \delta q_{i-1}$
$+ [D_2L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1L_d(\tilde{q}, q_i, (1-\alpha)h)] \cdot \delta\tilde{q}$
$+ [D_2L_d(\tilde{q}, q_i, (1-\alpha)h) + D_1L_d(q_i, q_{i+1}, h)] \cdot \delta q_i$
$+ [D_3L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3L_d(\tilde{q}, q_i, (1-\alpha)h)] \cdot h\delta\alpha$

where we have rearranged the summation and we have used the fact that $\delta q_0 = \delta q_N = 0$. This calculation is illustrated graphically in figure 1.

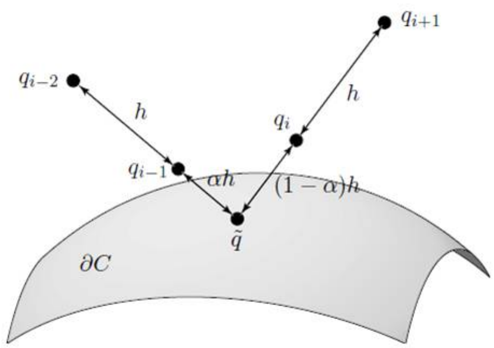

Fig. 2. The discrete variational principle for collisions

If we now require that the variations of the action be zero for any choice of δq_k, then we obtain the discrete Euler-Lagrangian equations

$$D_2L_d(q_{k-1}, q_k, h) + D_1L_d(q_k, q_{k+1}, h) = 0$$

which must hold for each k away from the impact time. For the particular L_d chosen above, we compute

$$D_2L_d(q_{k-1}, q_k, h) = M(\tfrac{q_k - q_{k-1}}{h})$$
$$D_1L_d(q_k, q_{k+1}, h) = -[M(\tfrac{q_{k+1} - q_k}{h}) + h\nabla V(q_k)]$$

and so the discrete Euler-Lagrange equations are

$$M(\tfrac{q_{k+1} - 2q_k + q_{k-1}}{h^2}) = -\nabla V(q_k)$$

This is clearly a discretization of Newton's equations, using a simple finite difference rule for the derivative. If we take initial conditions (q_0, q_1), then the discrete E-L equations define a recursive rule for calculating the sequence $\{q_k\}_{k=0}^N$. In this way, they define a map $F_{Ld}\colon (q_k, q_{k+1}) \mapsto (q_{k+1}, q_{k+2})$, which we can think of as a one-step integrator for the system defined by the continuous E-L equations, away from impact.

Near impact, we must consider the other equations which are implied by the discrete variation equation being zero. Assume that we have used the discrete E-L equations to compute the trajectory up until the pair (q_{i-2}, q_{i-1}), just before impact. Now we have the equation

$$D_2L_d(q_{i-2}, q_{i-1}, h) + D_1L_d(q_{i-1}, \tilde{q}, \alpha h) = 0$$

which becomes

$$M(\tfrac{\tilde{q} - q_{i-1}}{\alpha h}) - M(\tfrac{q_{i-1} - q_{i-2}}{h}) = -\alpha\nabla V(q_{i-1})$$

Combining this with the condition that $\tilde{q} \in \partial C$ we obtain $n+1$ equations to be solved for the $n+1$ unknowns $\tilde{q}$ and α. We thus now know the point and time of contact. Next, we recall that $\tilde{q} \in \partial C$ and so its variations must lie in the tangent space. This means that we have the pair of equations

$$[D_2L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1L_d(\tilde{q}, q_i, (1-\alpha)h)] \cdot \delta\tilde{q} = 0 \text{ for all } \delta\tilde{q} \in T\partial C$$
$$D_3L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3L_d(\tilde{q}, q_i, (1-\alpha)h) = 0$$

which becomes

$$M(\tfrac{q_i - \tilde{q}}{(1-\alpha)h}) - M(\tfrac{\tilde{q} - q_{i-1}}{\alpha h}) + (1-\alpha)h\nabla V(\tilde{q}) \in N_C(\tilde{q}), \text{———}(3)$$
$$[\tfrac{1}{2}(\tfrac{q_i - \tilde{q}}{(1-\alpha)h})^T M(\tfrac{q_i - \tilde{q}}{(1-\alpha)h}) + V(\tilde{q})]$$
$$-[\tfrac{1}{2}(\tfrac{\tilde{q} - q_{i-1}}{\alpha h})^T M(\tfrac{\tilde{q} - q_{i-1}}{\alpha h}) + V(q_{i-1})] = 0 - - - - - - -(4)$$

These are discrete versions of (1) and (2), and they give n equations to be solved for q_i. Finally, we use the equation,

$$D_2L_d(\tilde{q}, q_i, (1-\alpha)h) + D_1L_d(q_i, q_{i+1}, h) = 0,$$
which is
$$M(\tfrac{q_{i+1} - q_i}{h}) - M(\tfrac{q_i - \tilde{q}}{(1-\alpha)h}) = -\nabla V(q_i)$$

to solve for q_{i+1}, and then we can revert to using the standard discrete E-L equations to continue away from the impact.

The power of the variational approach becomes apparent when we consider the geometric properties of the discrete system. Just as in the continuous case, we can derive conservation laws of the discrete system directly from the variational principle. In particular, we will see that there is a conserved discrete symplectic form, and conserved discrete momentum maps aries from symmetries.

The central idea in discrete mechanics theory is the derivation of discrete time equations of motion for mechanical systems through discrete variational principles. This is in contrast to the common approach of obtaining discrete time equations of motion with a direct discretization of differential equations (using finite differences, quadrature rules, etc.).These includes free systems and systems with holonomic constraints. Also we can go for Discrete Mechanics and Optimal Control for the dynamical behavior of robot. This will be studies in our next paper.

IV. Conclusion and Future Directions

We have used variational principles to derive governing equations for a variety of impact behaviors. Furthermore, we have seen that how discrete system equations can be derived from continuous system equations in variational approach. One can find out discrete time equations of motion provide variational integration algorithms and act as constrains in the discrete mechanics and optimal control (DMOC) method for determining optimal controls.

Beyond this, future work may involve more detailed descriptions of contact release conditions, tools from the measure differential inclusion formulation of contact mechanics, or impact models involving multiple contacts.

Acknowledgment

We would like to thank Dr. N. M. Singh, Professor in Electrical Engineering Department of Veermata Jijabai Technological Institute (VJTI), Mumbai for his guidance and support throughout the paper.

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