جامعة جدة
University of Jeddah
Faculty of Engineering
Department of Electrical & Computer Engineering

Control Systems (ECE 331)

Mathematical Modeling of Physical Systems - III
(Modeling in the Frequency Domain)

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Modeling in Frequency Domain:

Laplace Transform Table:

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s + a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

Modeling in Frequency Domain: conti...

Laplace Transform Theorems:

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Case-I: Roots of the Denominator of F(s) are real & distinct

Example:

$$F(s) = \frac{2}{(s+1)(s+2)}$$

The roots of the denominator are distinct, since each factor is raised only to unity power.

Case-II: Roots of the Denominator of F(s) are real & repeated

Example:

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

The roots of the $(s+2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a multiple root of multiplicity 2.

Case-III: Roots of the Denominator of F(s) are complex or imaginary

Example:

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

This shows that denominator having complex roots.

Finding out Transfer Function:

Case-I: For Differential Equation

Problem: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

The transfer function is defined by $G(s)$ is:

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

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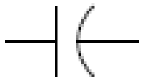
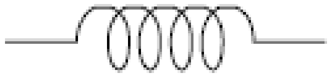
Finding out Transfer Function: conti…

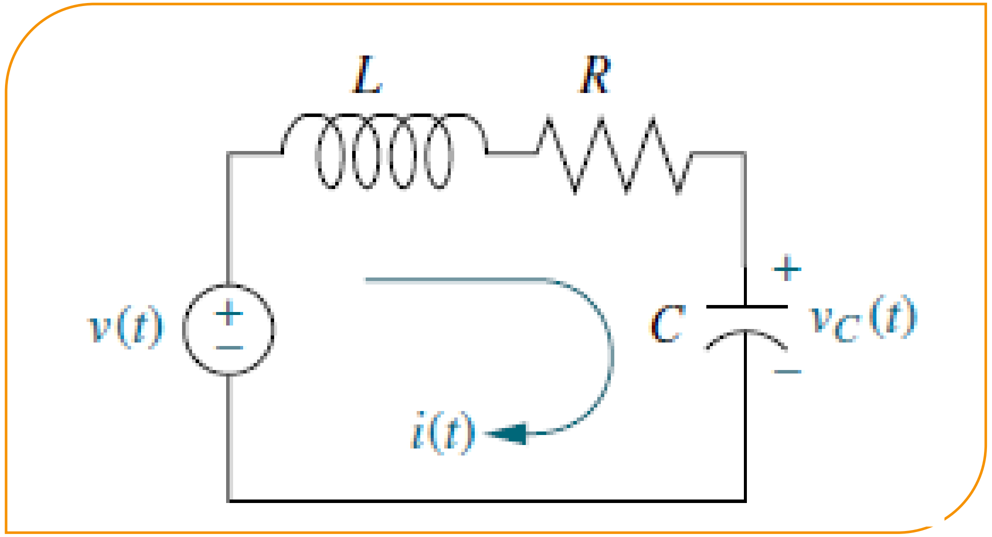
Case-II: For Electrical Network

Be remember the following topologies, while finding out transfer function

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Case-II: For Electric Network



Problem: Find transfer function relating the Capacitor voltage $Vc(s)$ to the input voltage $V(s)$ as shown in figure.

Solution: Summing the voltages around the loop, assuming zero initial conditions, gives the integral Differential equation for this network as >>

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\,i(\tau)d\tau=v(t)$$

Changing variables from current to change using $i(t)=\frac{dq}{dt}$ gives,

$$L\frac{d^2q(t)}{dt^2}+\mathrm{R}\,\frac{dq(t)}{dt}+\frac{1}{C}\,q(t)=v(t)$$

Finding out Transfer Function: conti…

Case-II: For Electric Network

From the voltage-charge relationship for a capacitor, we can write

$$q(t) = Cvc(t)$$

$$LC\frac{d^2vc(t)}{dt^2} + \mathrm{RC}\,\frac{dvc(t)}{dt} + vc(t) = v(t)$$

Taking Laplace of it,

$$LCs^2Vc(s) + \mathrm{RC\,s\,Vc(s)} + Vc(s) = V(s)$$

it gives ➡

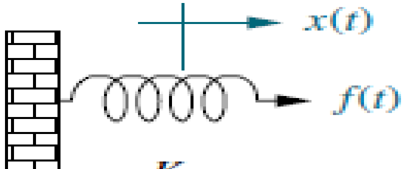
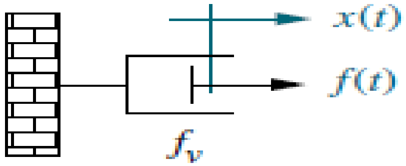
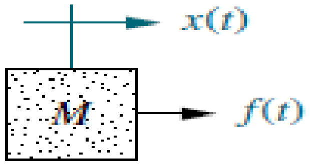
$$Transfer\ Function$$

$$\frac{Vc(s)}{V(s)} = \frac{1}{(\,LCs^2 + RCs + 1)}$$

Finding out Transfer Function: conti...

Case-III: For Translation Mechanical System

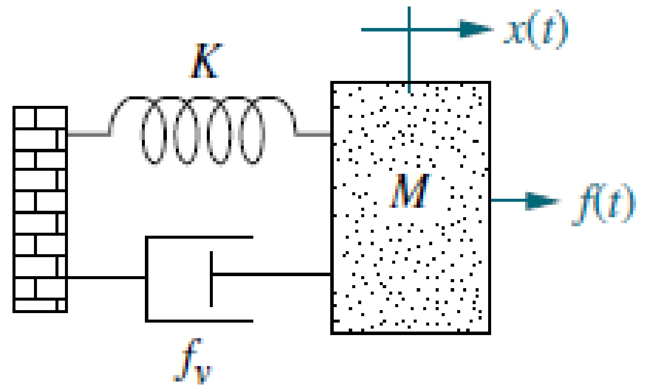
Be remember the following topologies, while finding out transfer function

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring  K	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper  f_v	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Case-III: For Translation Mechanical System

Problem: Find transfer function.

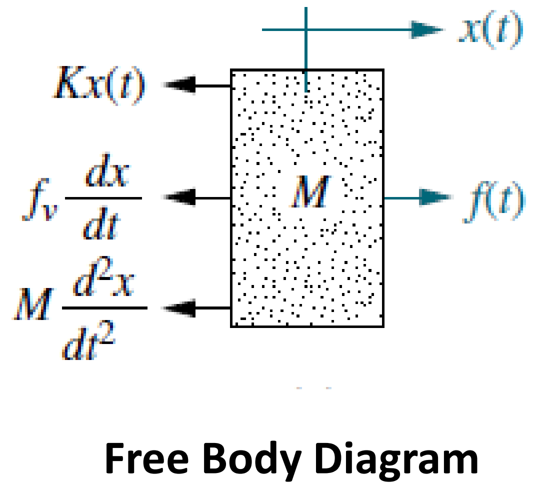


Solution: Using Newton’s law, we can write

$$M\frac{d^2x\,(t)}{dt^2} + \ f_v\ \frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking Laplace transform,

$$Ms^2X(s) + \ f_v\, sX(s) + KX(s) = F(s)$$

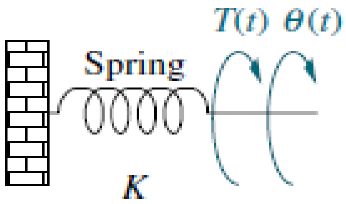
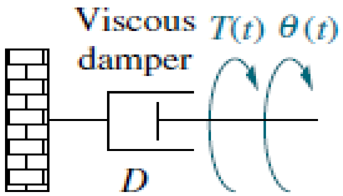
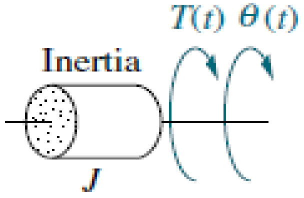


Transfer Function of the system is,

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + \ f_v\, s + K)}$$

Case-IV: For Rotational Mechanical System

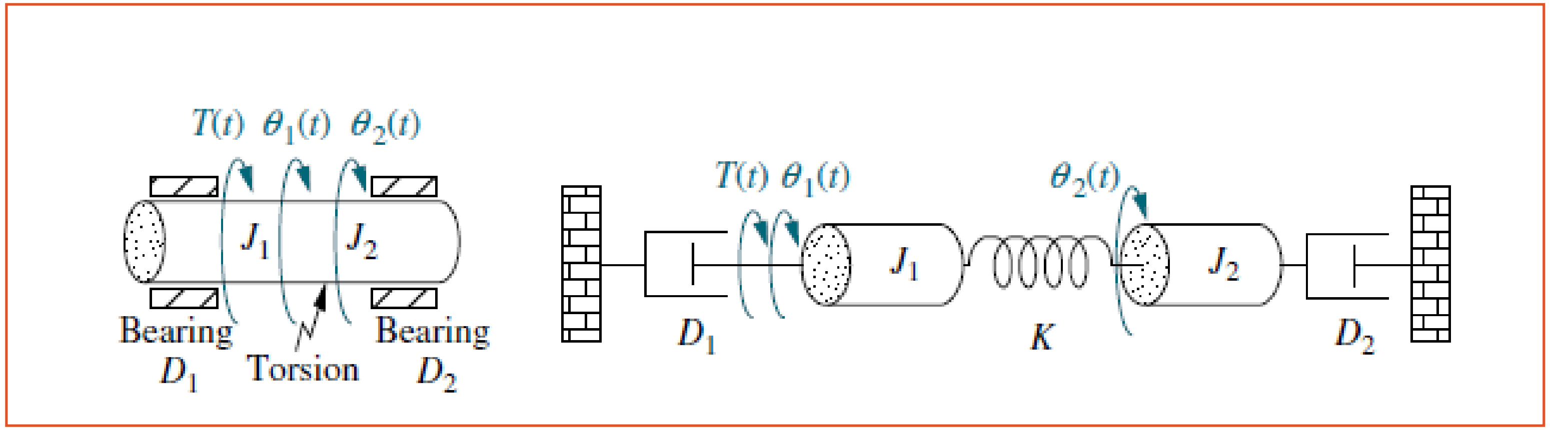
Be remember the following topologies, while finding out transfer function

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

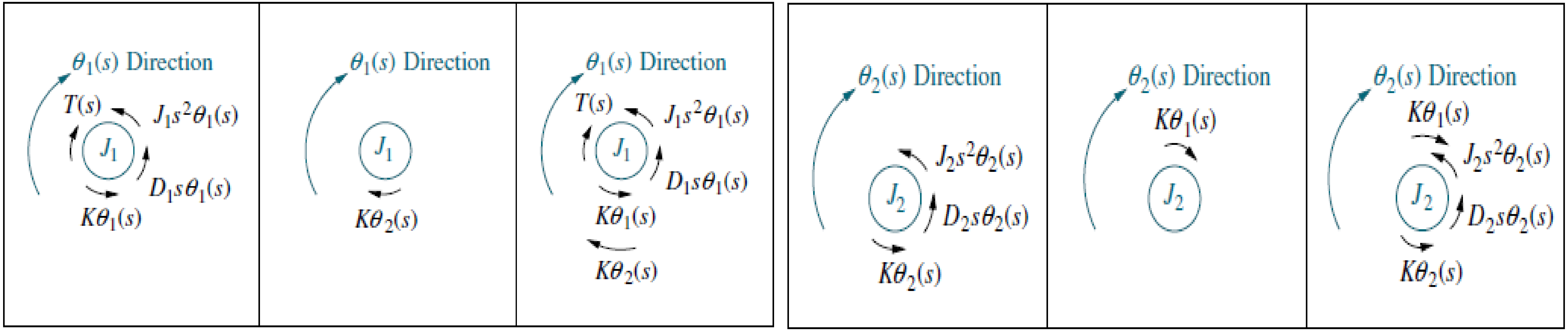
Case-III: For Rotational Mechanical System

Problem: Find transfer function.



Solution:

Free Body Diagram for J_1 & J_2



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Finding out Transfer Function: conti...

Case-III: For Rotational Mechanical System

Summing torques respectively for J_1 & J_2 , we obtain the equation of motion as:

$$(J_1 s^2 + D_1 s + K)\theta_1 s - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

The transfer function is:

$$G(s) = \frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

where, Δ is

$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

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Thank You !