University of Jeddah
Faculty of Engineering
Department of Electrical & Computer Engineering

Control Systems (ECE 331)

Mathematical Modeling of Physical Systems - II

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Differential Equations:

There are two types of differential equations:

Ordinary Differential Equations:

When the unknown function depends on a single independent variable, only ordinary derivatives appear in the equation. In this case, the equation is said to be an Ordinary Differential Equation (ODE).

Example:

$$\frac{dv}{dx} = 9.8 - 0.2v$$

Partial Differential Equations:

When the unknown function depends on several independent variables, partial derivatives appear in the equation. In this case, the equation is said to be an Partial Differential Equation (PDE).

Example:

$$\propto^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t} \quad \text{(Heat Equation)}$$

Laplace Transform:

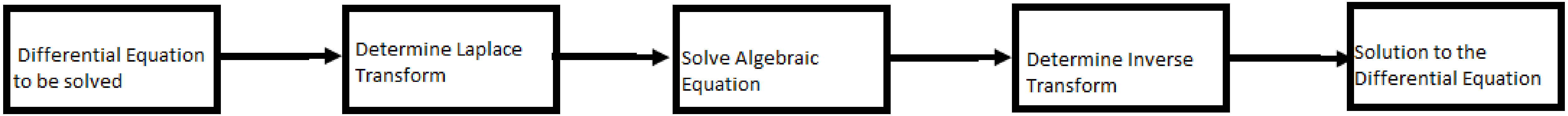
Pierre-Simon de Laplace (1749-1827), France.

Laplace transform is the operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

I. The given hard problem is transformed into a simple equation.
II. The simple equation is solved by purely algebraic manipulations.
III. The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.

Laplace Transform: conti...

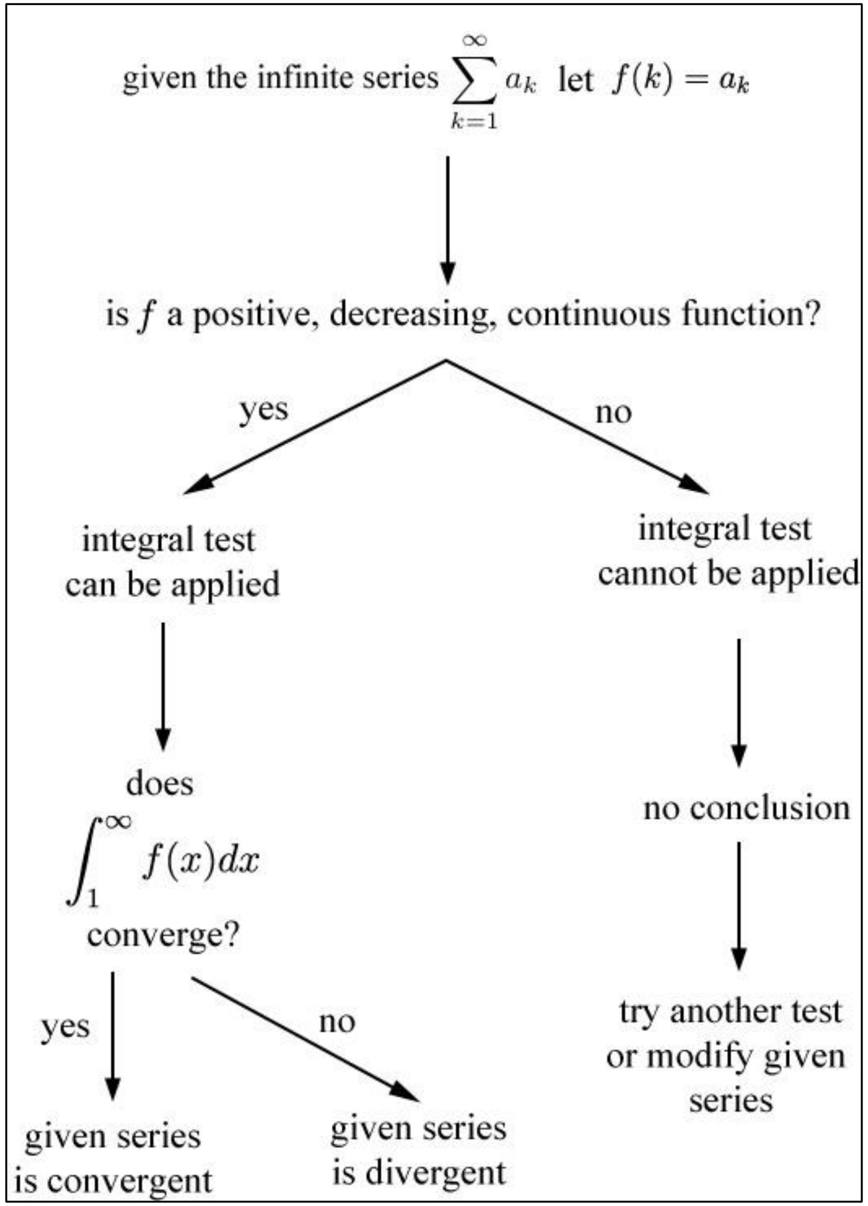


How Laplace Transform Works !!!

The Laplace transform is defined in the following way. Let *f(t)* be defined for $t \geq 0$. Then the Laplace transform of *f*, which is denoted by $\mathcal{L}\,[f(t)]$ or by F(s), is defined by the following equation:

$$\mathcal{L}\,[f(t)] = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defined a Laplace transform is an improper integral. An improper integral may converge or diverge, depending on the integrand. When the improper integral in convergent then we say that the function *f(t)* possesses a Laplace transform. So, what type of function possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.



Transfer Function:

“The Transfer Function of a linear, time-invariant, differential equations system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.”

Consider the linear time-invariant system defined as under:

$$a_0\, y^{(n)} + a_1\, y^{(n-1)} + \ldots\ldots\ldots + a_{n-1}\dot{y} + a_n y = b_0\, x^{(m)} + b_1\, x^{(m-1)} + \ldots\ldots\ldots + b_{m-1}\dot{x} + b_n x$$

where y = output of the system

x = input of the system

Transfer Function: conti…

The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero. It is written as:

$$\text{Transfer Function} = G(s) = \left.\frac{\mathcal{L}\,[Output]}{\mathcal{L}\,[Input]}\right|_{zero\ initial\ conditions}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in "*s*". If the highest power of "s" in the denominator of the transfer function is equal to "n", then the system is called an "nth order system".

Transfer Function: conti…

Properties of Transfer Function:

a. The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.

b. The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.

c. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system.

d. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

e. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

Convolution Integral:

For a linear time-invariant system the transfer function G(s) is : G(s) = $\frac{Y(s)}{X(s)}$

Where, X(s) is the Laplace transform of the input to the system and Y(s) is the Laplace transform of the output of the system, with assumption that all initial conditions are zero. Now, we can write above system in terms of Output form as:

$$Y(s) = G(s)\,X(s)$$

Inverse Laplace of the given equation as:

$$y(t) = \int_0^t x(\tau)\, g(t-\tau)d\tau$$

$$= \int_0^t g(\tau)x(t - \tau)d\tau$$; where both g(t) and x(t) are 0 for t<0.

Transfer Function: conti...

Impulse Response Function:

Consider the output (response) of a linear time invariant system to a unit-impulse input when the initial conditions are zero. Since the laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is:

$$Y(s) = G(s)$$

The inverse laplace transform of the output gives the impulse response of the system. The inverse Laplace transform of G(s)

$\mathcal{L}^{-1}[G(s)] = g(t)$ is called the impulse response function. This function g(t) is called the "weighting function of the system".

- The impulse response function g(t) is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The laplace transform of this function gives the transfer function. Therefore, the transfer function & impulse response function of a linear, time invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response.

Thank You !