University of Jeddah
Faculty of Engineering
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Electromagnetic Fields (ECE 308)

Lecture 3 – Vector Analysis - III

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Curl of a Vector Field:

The curl of a vector field **B** describes its rotational property or circulation. The circulation of **B** is defined as the line integral of **B** around a closed contour C;

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$

For rectangular contour abcd, as in fig.

$$\text{Circulation} = \int_a^b \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_b^c \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy + \int_c^d \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_d^a \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy$$

$$= B_0\, \Delta x - B_0\, \Delta x = 0,$$

According to above eq. the circulation of a uniform field is zero.

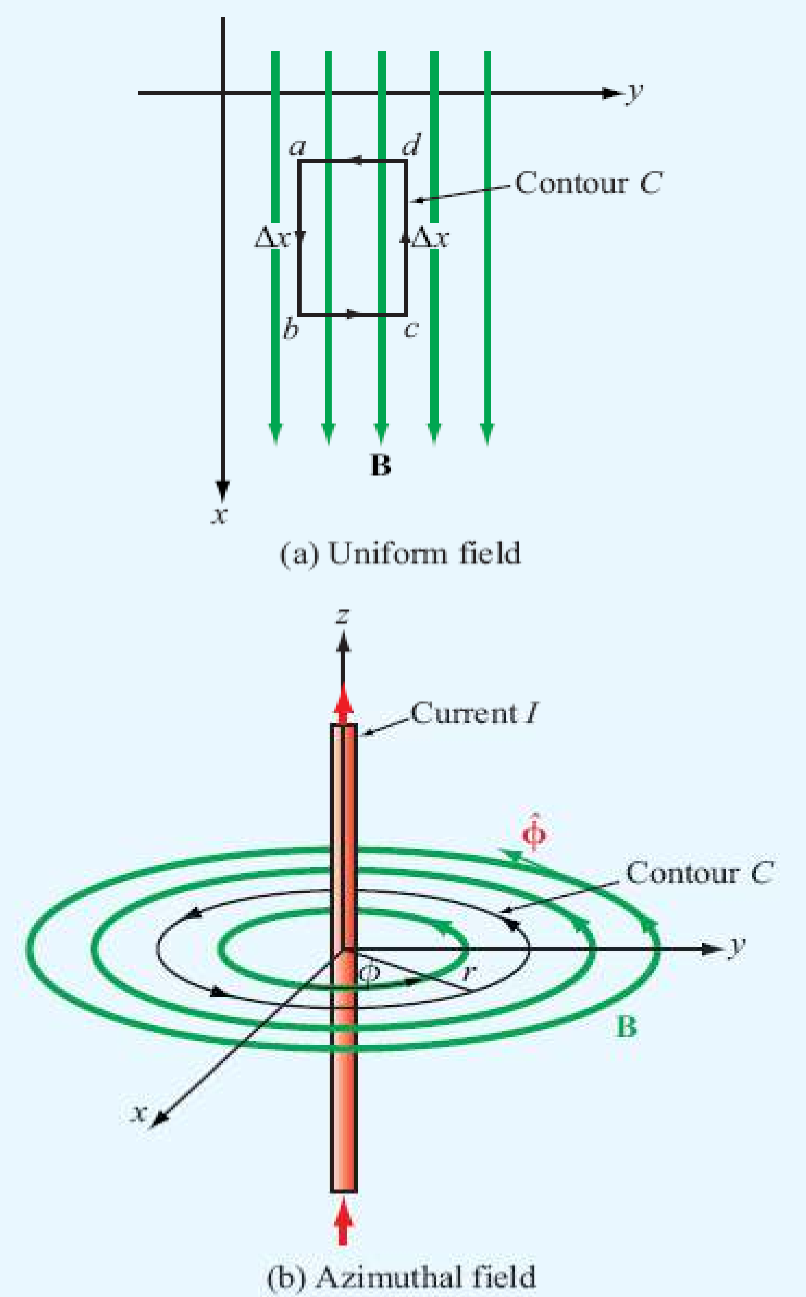

Figure 3-22 Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Curl of a Vector Field:

Now we consider magnetic flux density **B** induced by an infinite wire carrying a dc current I. If the current is in free space and it is oriented along the z direction, then,

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

- μ_0 is the permeability of free space.

- r is the radial distance from the current in the x-y plane.

The direction of B is along the azimuth unit vector ø. The field lines of B are concentric circles around the current.

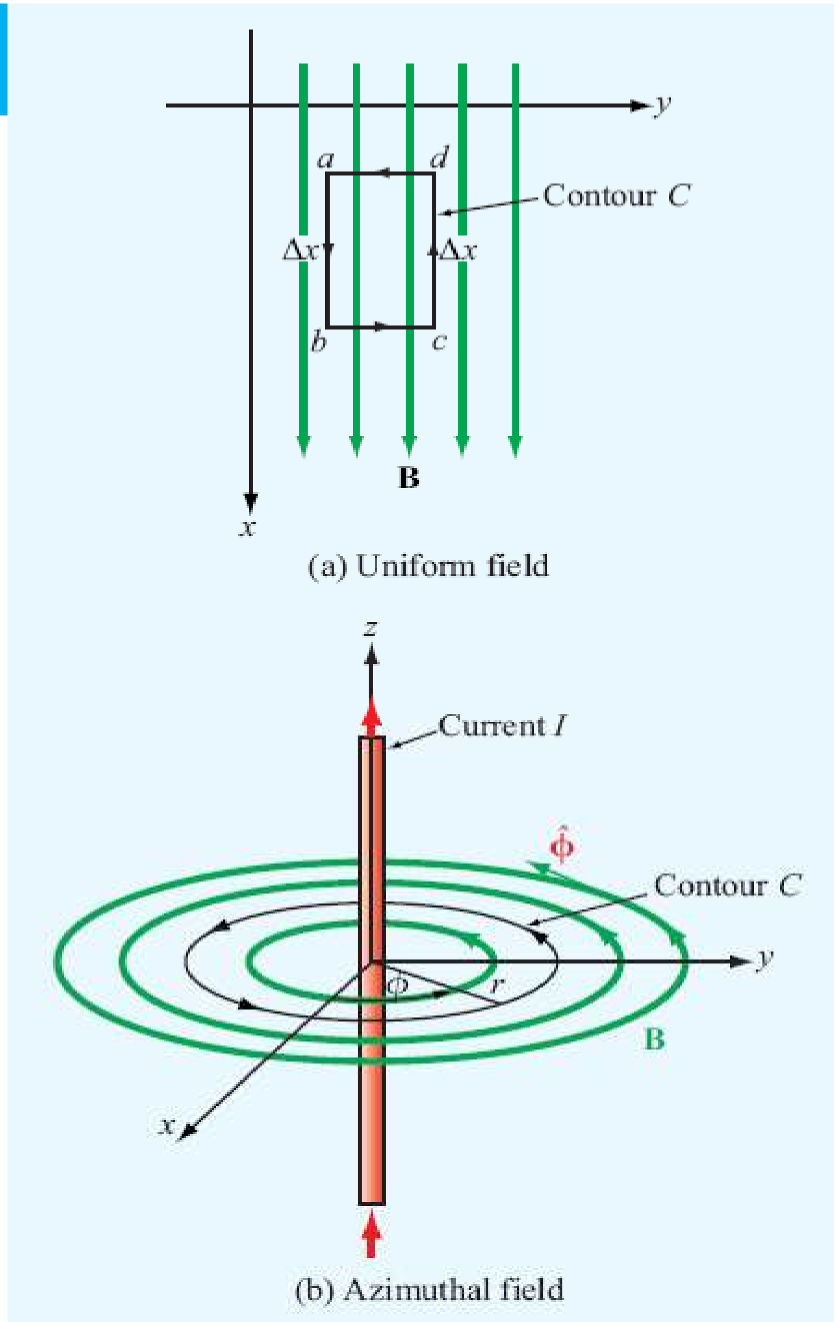

Figure 3-22 Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Curl of a Vector Field:

For a circular contour C of radius r centered at the origin in the x-y plane, the differential length vector $d\mathbf{l} = \hat{\boldsymbol{\phi}} r \, d\phi$, and the circulation of **B** is,

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

$$= \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r \, d\phi = \mu_0 I.$$

In this case, the circulation is not zero. However, had the contour C been in the x-z or y-z planes, dl would not have had a unit vector flux (ø) component, and the integral would have gives a zero circulation. It is clear that the circulation of **B** depends on the choice of contour and the direction in which it is traversed. For example: **Tornado**. We would like to choose our contour such that the circulation of the wind field is maximum, and we would like the circulation to have both a magnitude and a direction, with the direction being toward the tornado's vortex.

Curl of a Vector Field:

The curl of a vector field **B**, denoted **curl B** or $\nabla\times\mathbf{B}$ is defined as,

$$\nabla\times\mathbf{B} = \text{curl } \mathbf{B} = \lim_{\Delta s\to 0} \frac{1}{\Delta s}\left[\hat{\mathbf{n}} \oint_C \mathbf{B}\cdot d\mathbf{l}\right]_{\max} .$$

*"Curl **B** is the circulation of **B** per unit area, with the area Δs of the contour C being oriented such that the circulation is maximum."*

The direction of **Curl B** is defined by right hand rule.

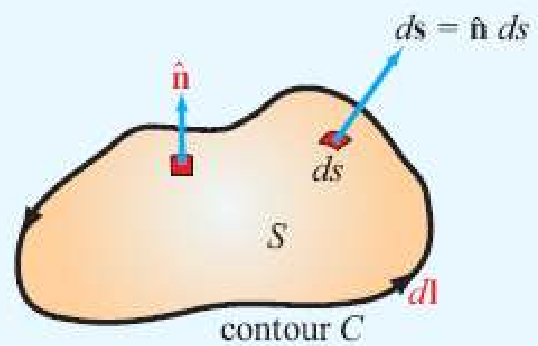

Figure 3-23 The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

For any two vectors **A** and **B** and scalar V,

(1) $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$,

(2) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$,

(3) $\nabla\times(\nabla V) = 0$.

Stokes's Theorem:

"Stokes's theorem converts the surface integral of the curl of a vector over an open surface C into a line integral of the vector along the contour C bounding the surface C."

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$

(Stokes's theorem)

If $\nabla \times \mathbf{B} = 0$ the field **B** is said to be Conservative **or** Irrotational, because its circulation, represented by the right hand side of above equation is zero, irrespective of the contour chosen.

Laplacian Operator:

We sometimes deal with problems involving multiple combinations of operations on scalars and vectors. A frequently encountered combination is the divergence of the gradient of a scalar. For a scalar function V defined in Cartesian coordinates, its gradient is:

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$= \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z = \mathbf{A},$$

Where we defined a vector A with components, Ax = ∂V / ∂x, Ay = ∂V / ∂y and Az = ∂V/ ∂z. The divergence of ∇V is

$$\nabla \cdot (\nabla V) = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

Laplacian Operator:

The term $\nabla \cdot (\nabla V)$ called the **Laplacian of V** and is denoted by $\nabla^2 V$ ("del square").

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

The Laplacian of a scalar can be used to define the Laplacian of a vector. For a vector **E**, specified in Cartesian coordinates, $\mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z$

The Laplacian of **E** is,

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} = \hat{\mathbf{x}} \nabla^2 E_x + \hat{\mathbf{y}} \nabla^2 E_y + \hat{\mathbf{z}} \nabla^2 E_z.$$

Thus, in Cartesian coordinates the Laplacian of a vector is a vector whose components are equal to the Laplacian of the vector components. It can be shown that,

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E})$$

Thank you !